SYDNEY
BOYS
HIGH
SCHOOL

NESA Number:

Name:

Maths Class: Circle

A B 1 2

S U M

2024

YEAR 12
TASK 4
TRIAL HSC

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- Solutions that rely totally on calculator technology may not necessarily be awarded full marks
- Unless otherwise stated, all answers should be left in simplified exact form
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 70

Section I – 10 marks (pages 2 – 5)

Attempt Questions 1 – 10
Allow about 15 minutes for this section

Section II – 60 marks (pages 6 – 13)

Attempt all Questions in Section II
Allow about 1 hour and 45 minutes for this section

Examiner:
AW

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–10.

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1 Let $p(x) = 2x^4 - 3x^3 - 5x^2 + 2x + 2$.

Given that the line $y = mx$, where m is real, crosses the curve $y = p(x)$ at four distinct points.

Let the x-coordinates of these points be x_1, x_2, x_3, and x_4.
What is the value of $x_1x_2x_3x_4$?

A. 0 B. 1 C. 2 D. 3

2 Which of the following is equivalent to $n(n-1)(n-2)\ldots(n-r+1)$?

A. $\dfrac{n!}{n-r!}$

B. $\dfrac{n!}{(n-r)!}$

C. $\dfrac{n!}{(n-r-1)!}$

D. $\dfrac{n!}{(n-r+1)!}$

3 Which of the following is equal to $\cos\alpha - \cos\beta$?

A. $2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

B. $2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$

C. $2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

D. $-2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$

4 Which of the following parametric equations represents a circle tangent to the y-axis?

A. $x = -1 + 2\cos\theta,\ y = -2 + 2\sin\theta$

B. $x = -2 + 2\cos\theta,\ y = -1 + 2\sin\theta$

C. $x = 1 + 2\cos\theta,\ y = 1 + 2\sin\theta$

D. $x = 2 + 3\cos\theta,\ y = 3 + 3\sin\theta$

5 A particle has a position vector given by $\mathbf{r} = 2\sin 3t\,\mathbf{i} - 2\cos 3t\,\mathbf{j}$ for $t \geq 0$.

What is the Cartesian equation of its path?

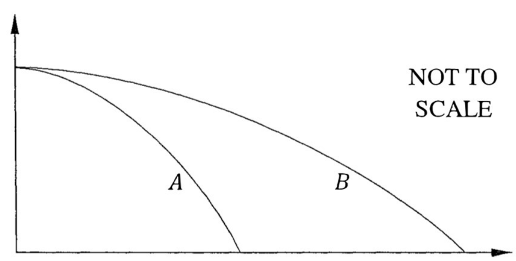
A. $x^2 + y^2 = 1, -2 \leq x \leq 2$

B. $y = \sqrt{4 - x^2}, -2 \leq x \leq 2$

C. $y = -\sqrt{4 - x^2}, -2 \leq x \leq 2$

D. $x^2 + y^2 = 4, -2 \leq x \leq 2$

6 Two balls, A and B, are rolled horizontally off a cliff at v m/s and $2v$ m/s respectively.



Which of the following statements is false?

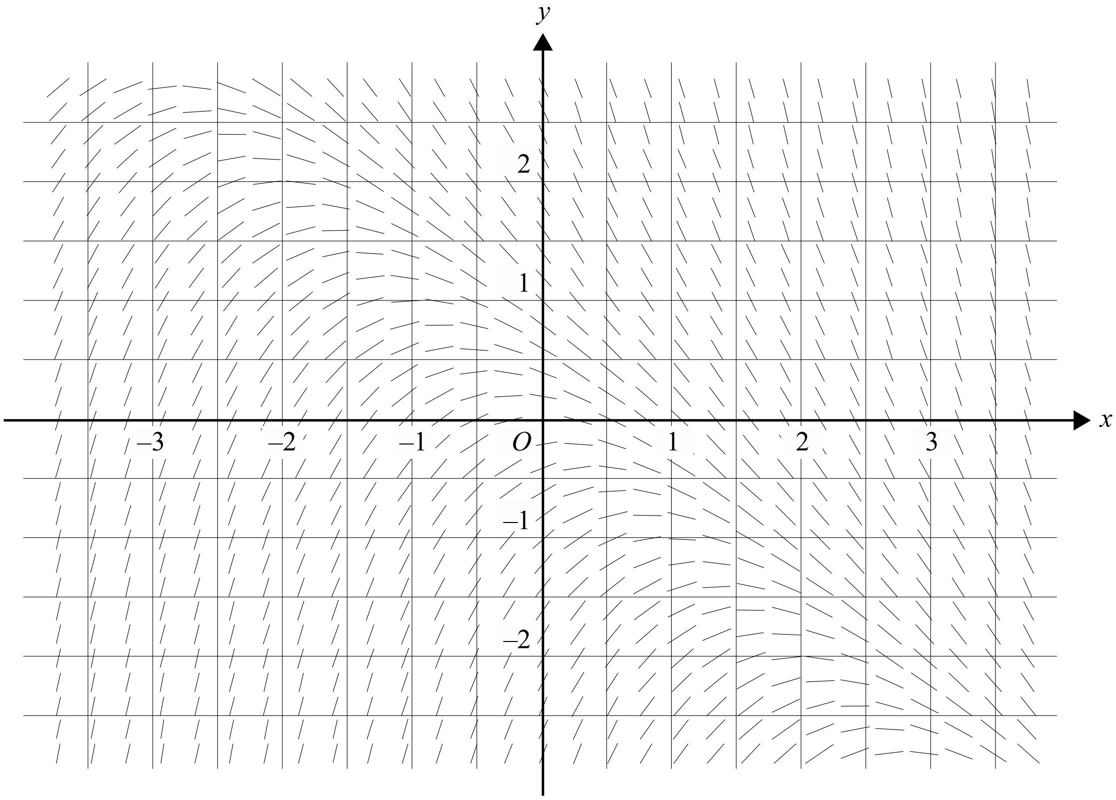
A. A and B are in the air for the same length of time.

B. A and B are travelling with the same vertical speed on impact.

C. B is travelling at twice the speed of A on impact with the ground.

D. B lands twice as far from the base of the cliff as A.

7



The direction field for the differential equation $\frac{dy}{dx} + x + y = 0$ is shown above.

A solution to this differential equation that includes (0, –1), would also include which of the following?

A. (3, –1)

B. (3.5, –2.6)

C. (–1.5, –2)

D. (2.6, –1)

8 The magnitudes of two vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are 3 and 2 respectively.

The angle between these two vectors is θ such that $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$.

Which of the following is the correct range of $\left| \underset{\sim}{p} - \underset{\sim}{q} \right|$?

A. $7 \le \left| \underset{\sim}{p} - \underset{\sim}{q} \right| \le 19$

B. $7 \le \left| \underset{\sim}{p} - \underset{\sim}{q} \right| \le 13$

C. $\sqrt{7} \le \left| \underset{\sim}{p} - \underset{\sim}{q} \right| \le \sqrt{19}$

D. $\sqrt{7} \le \left| \underset{\sim}{p} - \underset{\sim}{q} \right| \le 13$

9 Three standard, unbiased dice are tossed.
Given that the three uppermost faces have a sum of 8, what is the probability that exactly one of three dice has 4 on its uppermost face?

A. $\dfrac{4}{39}$

B. $\dfrac{3}{7}$

C. $\dfrac{5}{11}$

D. $\dfrac{6}{13}$

10 A tank initially holds 1000 L of water in which 10 kg of sugar is dissolved.

A solution containing 2 kg of sugar per litre flows into the tank at a rate of 6 L/min.

The mixture is stirred continuously and flows out of the tank at a rate of 9 L/min.

What is the differential equation for Q, the amount of sugar (in kilograms) in the tank after t minutes ?

A. $\dfrac{dQ}{dt} = 12 - \dfrac{9Q}{1000 - 3t}$

B. $\dfrac{dQ}{dt} = 12 - \dfrac{9Q}{1000 + 3t}$

C. $\dfrac{dQ}{dt} = 12 + \dfrac{9Q}{1000 + 3t}$

D. $\dfrac{dQ}{dt} = 12 + \dfrac{9Q}{1000 - 3t}$

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

In Questions 11-14, your responses should include ALL relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a SEPARATE writing booklet

(a) By first factorising $f(x)$, where $f(x) = -x^3 - 2x^2 + 4x + 8$, solve $\dfrac{1}{f(x)} < 0$. **3**

(b) State the value of n such that $\dbinom{n}{12} = \dbinom{n}{8}$. **1**

(c) (i) Prove by mathematical induction that for all $n \in \mathbb{Z}^+$, **3**

$$1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + n) = \frac{1}{6}n(n+1)(n+2).$$

(ii) Hence find, in terms of n **2**

$$3 + (3 + 6) + (3 + 6 + 9) + \ldots + (3 + 6 + 9 + \ldots + (6n - 3)).$$

(d) Explain why, that in any party with two or more people, there must be at least two people who have the same number of friends. **2**

You may assume that if X is friends with Y, then Y is friends with X.

Consider the case where everyone has at least one friend and the case where there is someone who has no friends.

Question 11 continues on page 7

(e) Consider the differential equation **3**

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{1 - \sin x},$$

where $0 \le x < \frac{\pi}{2}$ and $y > 0$.

Given that $y = 1$ when $x = \frac{\pi}{6}$, express y as a function of x.

End of Question 11

Question 12 (16 marks) Use a SEPARATE writing booklet

(a) The function f is defined by $f(x) = \dfrac{2x+a}{x-2}$, where a is a constant, $x > 2$.

(i) Given that f^{-1} exists, state the value that a cannot take, justifying your answer. **2**

(ii) A function h is said to be self-inverse if $h(x) = h^{-1}(x)$ for all x in the domain of h. **1**

State the range of values of a such that f is a self-inverse function.

(b) Let $\underset{\sim}{c} = |\underset{\sim}{a}|\underset{\sim}{b} + |\underset{\sim}{b}|\underset{\sim}{a}$, where $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$ are non-zero vectors. **3**

Show that $\underset{\sim}{c}$ bisects the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(c) The population of dingoes on an island is modelled by the logistic equation

$$\frac{dy}{dt} = y(1-y),$$

where y is the fraction of the island's carrying capacity of dingoes reached after t years.

Initially, the population of dingoes is estimated to be one-quarter of the island's carrying capacity.

(i) Use the substitution $y = \dfrac{1}{1-w}$ to transform the logistic equation to $\dfrac{dw}{dt} = -w$. **2**

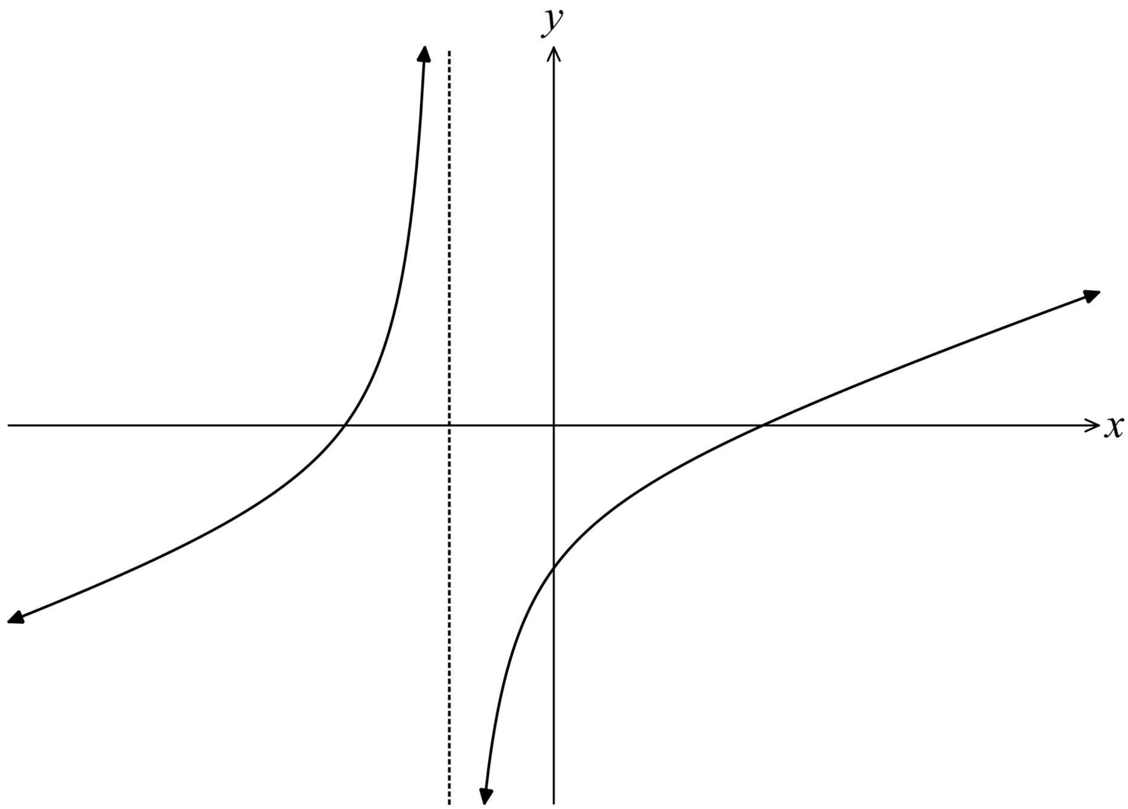
(ii) 1. Show that $w = Ae^{-t}$ is a solution to $\dfrac{dw}{dt} = -w$. **1**

2. Using the solution of $\dfrac{dw}{dt} = -w$, find the solution of the logistic equation for y satisfying the initial conditions. **2**

(iii) Show when $t = \ln 9$, the dingo population is three-quarters of the island's carrying capacity. **1**

Question 12 continues on page 9

(d) The graph of $y = \dfrac{x^2 - 4}{x + 1}$ is shown below.



Use the Answer sheet provided to answer the following questions.

Put your Answer Sheet for this question inside your Answer Booklet for Question 12.

(i) Sketch $y = \sqrt{f(x)}$. **2**

(ii) Sketch $y = \dfrac{|x^2 - 4|}{x + 1}$. **2**

End of Question 12

Question 13 (16 marks) Use a SEPARATE writing booklet

(a) Hugh has six pairs of socks in a drawer, each pair is of a different brand, including his favourite X-men brand.

Each pair consists of two identical socks.

He selects one sock at a time and at random, without replacement.

(i) What is the minimum number of socks he needs to pull out so that he has at least one matching pair socks? **1**

(ii) Find the least number of socks he must select so that the probability of having the X-men pair is higher than the probability of having only 1 X-men sock. **3**

(b) A ship S is travelling with a constant velocity, $\underset{\sim}{v}$ km/h, where

$$\underset{\sim}{v} = \begin{pmatrix} -12 \\ 15 \end{pmatrix}.$$

At time $t = 0$, the ship is at point $A\,(300, 100)$ relative to an origin O, where distances are measured in kilometres.

A lighthouse is located at a point $L\,(129, 283)$.

(i) Show that $\overrightarrow{LS} = \begin{pmatrix} 171 \\ -183 \end{pmatrix} + t\begin{pmatrix} -12 \\ 15 \end{pmatrix}$ **2**

(ii) By finding $\left|\overrightarrow{LS}\right|$, find the value of t when the ship is closest to the lighthouse. **2**

(iii) An alarm will sound if the ship travels within 20 km of the light house. State whether the alarm will sound. Justify your answer. **1**

Question 13 continues on page 11

Question 13 (continued)

(c) A vertical wall, height H metres, stands on horizontal ground.

An object is thrown towards the wall and is projected with an initial speed u m/s at an angle θ with the horizontal plane.

The object is starts from a point on the ground D metres from the wall and it just clears the wall at the highest point of its path.

Let the position of the object at time t be given by $\mathbf{r}(t)$ where

$$\mathbf{r}(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$$ (Do NOT prove this.)

(i) Show that the particle reaches the highest point on its path when $t = \dfrac{u\sin\theta}{g}$. **2**

(ii) Show that the speed of projection is given by $u^2 = \dfrac{g}{2H}\left(4H^2 + D^2\right)$. **3**

(iii) Show that θ, the angle of projection, is given by **2**

$$\theta = \tan^{-1}\left(\frac{2H}{D}\right).$$

End of Question 13

Question 14 (14 marks) Use a SEPARATE writing booklet

(a) With respect to the origin O, points A and B have position vectors **a** and **b** respectively.

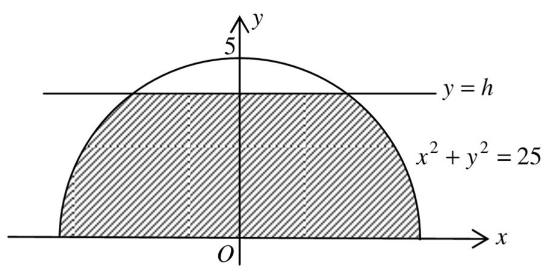
Point P is on the line AB such that $AP : PB = m : n$, where m and n are positive integers.

Point C is on OP extended such that $OP : PC = 1 : 2$.

(i) Show that $\overrightarrow{AC} = \left(\dfrac{2n-m}{m+n}\right)\mathbf{a} + \left(\dfrac{3m}{m+n}\right)\mathbf{b}$ **3**

(ii) Find the ratio $AP : PB$ such that AC is parallel to OB. **1**

(b) (i) **2**



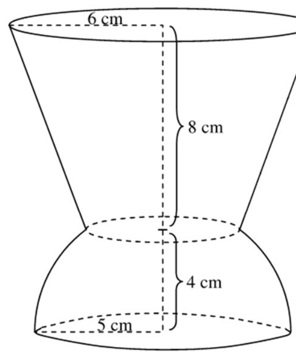
In the figure above, the shaded region enclosed by the circle $x^2 + y^2 = 25$, the x-axis, and the straight line $y = h$ (where $0 \le h \le 5$) is revolved about the y-axis.

Show that the volume of the solid of revolution is $\left(25h - \dfrac{h^3}{3}\right)\pi$ cubic units.

Question 14 continues on page 13

Question 14 (continued)

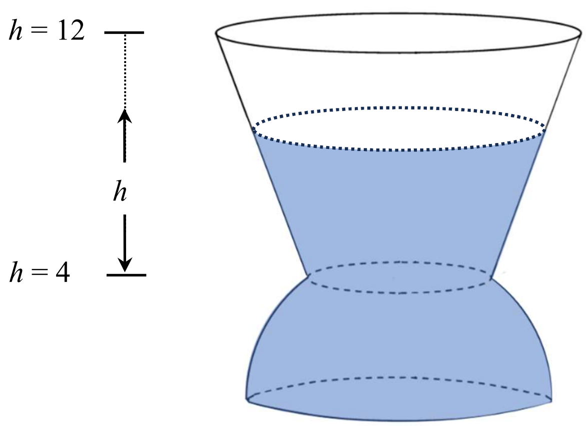
(b) (continued)
In the diagram below, an empty coffee cup consists of two parts.
The lower part is in the shape of the solid described in (b) (i).
The upper part is a frustum of a circular cone of height 8 cm, where the radius of the top of the cup is 6 cm.
Hot coffee is poured into the cup to a depth h cm at a rate of 8 cm^3/s, where $0 \le h \le 12$.
Let V cm^3 be the volume of coffee in the cup.



(ii) Find the rate of increase of the depth of coffee when the depth is 3 cm. **2**

(iii) Show that the volume of coffee in the cup when for $4 \le h \le 12$ is given by **3**

$$V = \frac{164\pi}{3} + \frac{3\pi}{64}(h+4)^3 .$$



(iv) After the cup is fully filled, it cracks at the bottom. **3**
The coffee leaks at a rate of 2 cm^3/s.

Find the rate of decrease of the depth of coffee after 15 seconds of leaking.
Leave your answer correct to 3 significant figures.

End of paper

SYDNEY BOYS HIGH SCHOOL

2024

YEAR 12

HSC TASK 4

Mathematics Extension 1

Sample Solutions

NOTE: This process of checking your mark is about reading the solutions and the comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.

Just because you have shown some 'working' does not justify that your solution is worth any marks.

Students who used pencil, an erasable pen and/or whiteout, may NOT be able to appeal.

MC Answers

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
B	B	D	B	D	C	B	C	B	A

Section I Multiple Choice Solutions

1 Let $p(x) = 2x^4 - 3x^3 - 5x^2 + 2x + 2$.

Given that the line $y = mx$, where m is real, crosses the curve $y = p(x)$ at four distinct points.

Let the x-coordinates of these points be x_1, x_2, x_3, and x_4.
What is the value of $x_1x_2x_3x_4$?

A. 0 B. 1 C. 2 D. 3

$2x^4 - 3x^3 - 5x^2 + 2x + 2 = mx \Rightarrow 2x^4 - 3x^3 - 5x^2 + (2-m)x + 2 = 0$

A	9
B	125
C	29
D	0

2 Which of the following is equivalent to $n(n-1)(n-2)\ldots(n-r+1)$?

A. $\dfrac{n!}{n-r!}$ B. $\dfrac{n!}{(n-r)!}$

C. $\dfrac{n!}{(n-r-1)!}$ D. $\dfrac{n!}{(n-r+1)!}$

A	8
B	126
C	17
D	12

3 Which of the following is equal to $\cos\alpha - \cos\beta$?

A. $2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

B. $2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$

C. $2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

D. $-2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$

Using the Reference sheet, options A and C are out.

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

A	5
B	36
C	8
D	114

4 Which of the following parametric equations represents a circle tangent to the y-axis?

A. $x = -1 + 2\cos\theta,\ y = -2 + 2\sin\theta$

B. $x = -2 + 2\cos\theta,\ y = -1 + 2\sin\theta$

C. $x = 1 + 2\cos\theta,\ y = 1 + 2\sin\theta$

D. $x = 2 + 3\cos\theta,\ y = 3 + 3\sin\theta$

A	26
B	100
C	24
D	13

The radius must equal the distance to the y-axis, which would be the x-coordinate of the centre.

5 A particle has a position vector given by $\mathbf{r} = 2\sin 3t\,\mathbf{i} - 2\cos 3t\,\mathbf{j}$ for $t \geq 0$.

What is the Cartesian equation of its path?

A. $x^2 + y^2 = 1,\ -2 \leq x \leq 2$

B. $y = \sqrt{4 - x^2},\ -2 \leq x \leq 2$

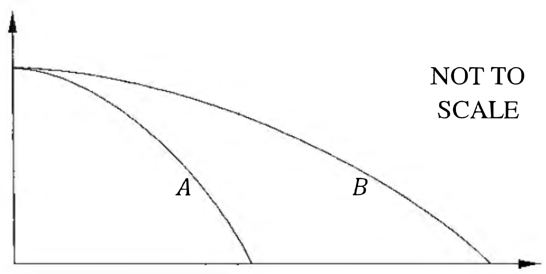
C. $y = -\sqrt{4 - x^2},\ -2 \leq x \leq 2$

D. $x^2 + y^2 = 4,\ -2 \leq x \leq 2$

A	2
B	21
C	50
D	89

Option A is clearly out. Options B and C are out because $-2\cos 3t$ can be pos./neg.

6 Two balls, A and B, are rolled horizontally off a cliff at v m/s and $2v$ m/s respectively.



A	18
B	23
C	109
D	13

Which of the following statements is false?

A. A and B are in the air for the same length of time.

B. A and B are travelling with the same vertical speed on impact.

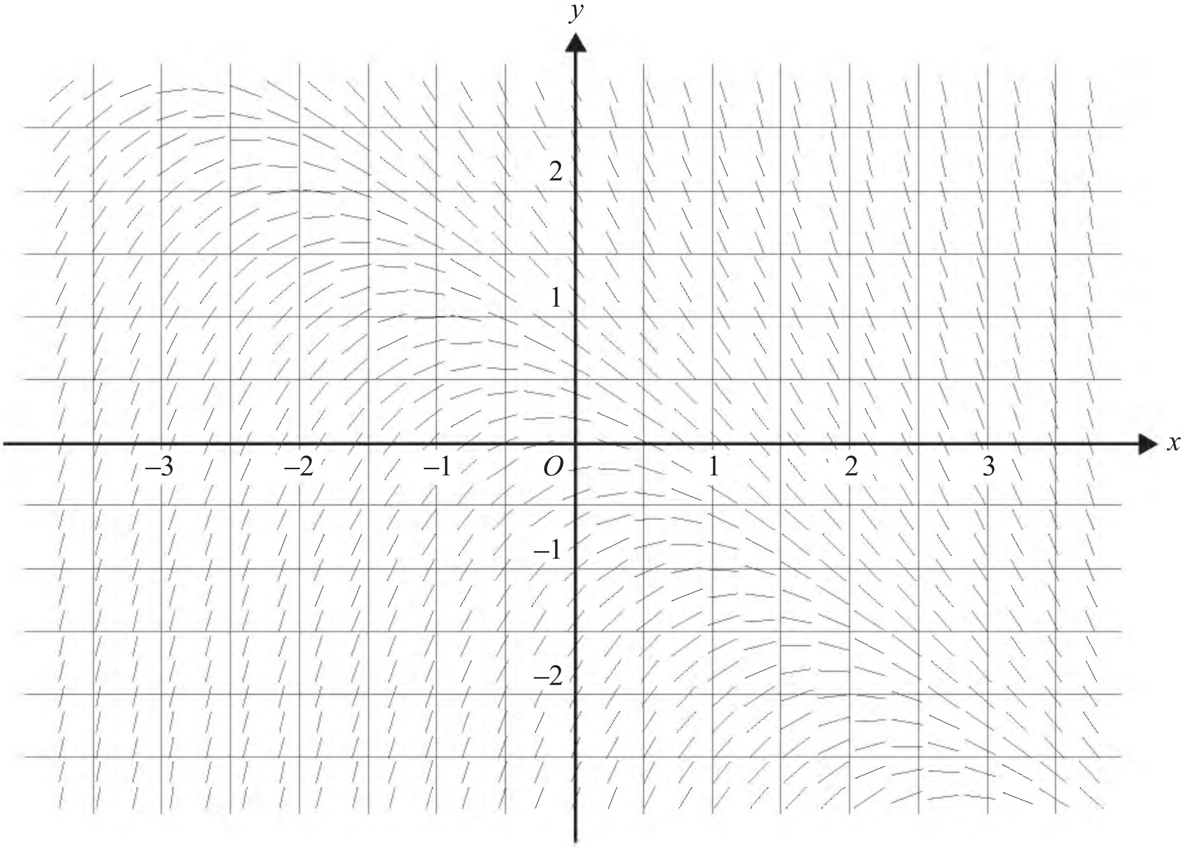
C. B is travelling at twice the speed of A on impact with the ground.

D. B lands twice as far from the base of the cliff as A.

Options A, B, & D are facts that you need to know.

You also need to differentiate between questions that ask for ‘velocity’ vs ‘vertical velocity’.

7



The direction field for the differential equation $\frac{dy}{dx} + x + y = 0$ is shown above.

A solution to this differential equation that includes (0, –1), would also include which of the following?

A. (3, –1)

B. (3.5, –2.6)

C. (–1.5, –2)

D. (2.6, –1)

Options A & D are out as they are on the 'other side' of $y = x + 1$.
Option C is out because the 'flow' or 'current' would not take it that far.

A	4
B	145
C	11
D	3

8 The magnitudes of two vectors p and q are 3 and 2 respectively.

The angle between these two vectors is θ such that $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$.

Which of the following is the correct range of $\left| \underset{\sim}{p} - \underset{\sim}{q} \right|$?

A	18
B	12
C	111
D	22

A. $7 \le \left| \underset{\sim}{p} - \underset{\sim}{q} \right| \le 19$

B. $7 \le \left| \underset{\sim}{p} - \underset{\sim}{q} \right| \le 13$

C. $\sqrt{7} \le \left| \underset{\sim}{p} - \underset{\sim}{q} \right| \le \sqrt{19}$

D. $\sqrt{7} \le \left| \underset{\sim}{p} - \underset{\sim}{q} \right| \le 13$

Using the cosine rule:

$$|\mathbf{p} - \mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2 - 2 \times |\mathbf{p}| \times |\mathbf{q}| \cos\theta$$

$$= 9 + 4 - 2 \times 3 \times 2 \cos\theta$$

$$= 13 - 12 \cos\theta$$

Now for $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$, $\cos\theta$ is a strictly decreasing function

$-\frac{1}{2} \le \cos\theta \le \frac{1}{2} \Rightarrow -6 \le -12\cos\theta \le 6$

$\therefore\ 7 \le 13 - 12\cos\theta \le 19 \Rightarrow 7 \le |\mathbf{p} - \mathbf{q}|^2 \le 19$

$\therefore\ \sqrt{7} \le |\mathbf{p} - \mathbf{q}| \le \sqrt{19}$

9 Three standard, unbiased dice are tossed.
Given that the three uppermost faces have a sum of 8, what is the probability that exactly one of three dice has 4 on its uppermost face?

A	27
B	81
C	39
D	16

A. $\frac{4}{39}$

B. $\frac{3}{7}$

C. $\frac{5}{11}$

D. $\frac{6}{13}$

Adding to 8

Cases:	(1, 1, 6)	There are 3 of these.
	(1, 2, 5)	There are 6 of these.
	(1, 3, 4)	There are 6 of these.
	(2, 2, 4)	There are 3 of these.
	(3, 3, 2)	There are 3 of these.
Total:	3 + 6 + 6 +3 + 3 = 21	
Exactly one 4	(1, 3, 4) & (2, 2, 4)	There are 9 of these.

10 A tank initially holds 1000 L of water in which 10 kg of sugar is dissolved.

A solution containing 2 kg of sugar/litre flows into the tank at a rate of 6 L/min.

The mixture is stirred continuously and flows out of the tank at a rate of 9 L/min.

What is the differential equation for Q, the amount of sugar (in kilograms) in the tank after t minutes ?

A. $\dfrac{dQ}{dt} = 12 - \dfrac{9Q}{1000 - 3t}$

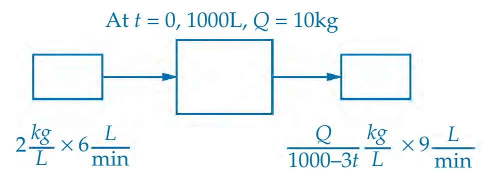
B. $\dfrac{dQ}{dt} = 12 - \dfrac{9Q}{1000 + 3t}$

A. $\dfrac{dQ}{dt} = 12 + \dfrac{9Q}{1000 + 3t}$

A. $\dfrac{dQ}{dt} = 12 + \dfrac{9Q}{1000 - 3t}$

A	81
B	53
C	11
D	17

$$\frac{dQ}{dt} = \frac{dQ}{dV} \times \frac{dV}{dt}$$



We use $\dfrac{Q}{1000-3t}$, since the original volume is 1000 where less liquid is in the tank for every unit of time. In this case, this is $6 - 9 = -3$ units every minute.

This gives

$$\begin{aligned} \frac{dQ}{dt} &= \text{inflow} - \text{outflow} \\ &= (2 \times 6) - \left(\frac{Q}{1000-3t} \times 9\right) \\ &= 12 - \frac{9Q}{1000-3t} \end{aligned}$$

End of Section I Solution

Section II Question 11 Solutions 14 Marks

(a) By first factorising $f(x)$, where $f(x) = -x^3 - 2x^2 + 4x + 8$, solve $\dfrac{1}{f(x)} < 0$. **3**

$$f(x) = -x^2(x+2) + 4(x+2)$$
$$= (x+2)(4-x^2)$$
$$= (x+2)^2(2-x)$$

In $\dfrac{1}{f(x)} = \dfrac{1}{(x+2)^2(2-x)}$, $(x+2)^2 > 0$

$\therefore \dfrac{1}{f(x)}$ has the same sign as $(2-x)$

If $\dfrac{1}{f(x)} < 0$

$2 - x < 0$

$\therefore x > 2$

Well done.

Many students only partially factorised the expression, no mark was deducted this time. However, please remember that always completely factorise the expression.

This is a 3 mark questions, students are expected to justify their answers.

Graphical method was accepted as well.

(b) State the value of n such that $\dbinom{n}{12} = \dbinom{n}{8}$. **1**

${}^nC_r = {}^nC_{n-r}$

$\therefore n = 12 + 8 = 20$

Well done.

No half marks

(c) (i) Prove by mathematical induction that for all $n \in \mathbb{Z}^+$, **3**

$$1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + n) = \frac{1}{6}n(n+1)(n+2).$$

Prove that the statement is true for $n = 1$.

When $n = 1$,

$\text{LHS} = 1$

$\text{RHS} = \frac{1}{6} \times 1 \times 2 \times 3 = 1$

$\therefore$ LHS = RHS

$\therefore$ The statement is true for $n = 1$

Assume the statement is true for $n = k$, $k \in \mathbb{Z}^+$,

i.e. $1 + (1+2) + (1+2+3) + \cdots + (1+2+3+\cdots+k) = \frac{1}{6}k(k+1)(k+2)$

Prove the statement is true for $n = k+1$,

i.e. $1 + (1+2) + (1+2+3) + \cdots + (1+2+3+\cdots+k+k+1) = \frac{1}{6}(k+1)(k+2)(k+3)$

$$\begin{aligned}\text{LHS} &= \frac{1}{6}k(k+1)(k+2) + (1+2+3+\cdots+k) \\ &= \frac{1}{6}k(k+1)(k+2) + \frac{(k+1)(k+2)}{2} \\ &= \frac{1}{6}(k+1)(k+2)(k+3) \\ &= \text{RHS}\end{aligned}$$

Therefore, by mathematical induction, the statement is true for all $n \in \mathbb{Z}^+$.

Extremely well done.

Common errors:

- *Incorrect format of Proof questions*
- *Some students did not recognise* $(1+2+3+\cdots+k+k+1)$ *is a simple AP.*

(c) (ii) Hence find, in terms of n **2**

$$3 + (3 + 6) + (3 + 6 + 9) + \ldots + (3 + 6 + 9 + \ldots + (6n - 3)).$$

$$3+(3+6)+(3+6+9)+\cdots+\left(3+6+9+\cdots(6n-3)\right)=3\left[1+(1+2)+(1+2+3)+\cdots+\left(1+2+3+\cdots(2n-1)\right)\right]$$

$$=3\times\frac{1}{6}\times(2n-1)(2n-1+1)(2n-1+2)$$

$$=n(2n-1)(2n+1)$$

Poorly done. Some students did not attempt this question.

Many students were not able to recognise that $k=2n-1$

Did not simplify their final answers

Few students did not multiple their answer by 3.

Common wrong answers: $\frac{1}{2}(2n-1)(2n)(2n+1)$, $\frac{1}{6}(2n-1)(2n)(2n+1)$,

$\frac{1}{2}n(n-1)(n+1)$, $\frac{1}{2}(6n-1)(6n)(6n+1)$

(d) Explain why, that in any party with two or more people, there must be at least two people who have the same number of friends. **2**

You may assume that if X is friends with Y, then Y is friends with X.

Consider the case where everyone has at least one friend and the case where there is someone who has no friends.

Case 1: *everyone has at least one friend*

If everyone has at least one friend, then each person has between 1 to $n-1$ friends. Each of the n partygoers can be categorised as one of these $n-1$ values and hence two of the partygoers must have the same value. That is, the same number of friends by the pigeonhole principle.

Case 2: *someone has no friends*

If someone lacks any friends, then that person is a stranger to all other guests – a singleton. Because 'friend' is symmetric, the highest value anyone else could have is $n-2$, that is, they would be friends with everyone except the singleton. Therefore, everyone has between 0 to $n-2$ friends.
This means of the n partygoers can be categorised as one of the $n-1$ values, and hence two of the partygoers must have the same value, or number of friends.

Students' examples:

d) Consider n people ($n \in \mathbb{Z}$, $n \geq 2$)

Case 1: everyone has at least one friend

$\Rightarrow$ number of friends someone can have $= 1, 2, 3 \ldots (n-1)$ (excludes themselves) ✓

Case 2: someone ~~[illegible]~~ has no friends

$\Rightarrow$ number of friends someone can have $= 0, 1, 2, 3 \ldots (n-2)$

$(n-2)$ since they cannot be friends with the person with no friends ✓

Cases 1 and 2 are mutually exclusive

In each, there are n pigeons (n people) and $(n-1)$ pigeonholes (number of friends someone can have)

$\therefore$ by pigeonhole principle, at least two people have same number of friends ✓

d) If all have at least 1 friend:
n people total
Let the first $(n-1)$ people have diff. amount of friends.
Best case: $1, 2, 3, 4, \ldots\ n-1$
The last person must be friends with at least 1
max $(n-1)$ since there are $(n-1)$ other people
But no matter what, he will have the sa number as someone from the $(n-1)$ people.
$\therefore$ must be at least 2 people with same number of friends.

If someone has no friends:
~~[illegible]~~ n people total
There are $(n-1)$ people with at least 1 friend.
Similarly to the first case, the first $(n-2)$ people have $1, 2, 3, \ldots, n-2$ friends (best case), but the last person can only have from 1 to $n-2$ friends which is the same as someone from the $(n-2)$ people.
$\therefore$ In both cases, at least 2 people with same no. of friends.

Extremely poorly done.

Many students did not attempt this question.

No mark was awarded if students only had the pigeonhole principle as the answer.

Common errors:

- *Some students only explained one case.*
- *Many students only discussed the question when the party has 2 or 3 people and did not provide a general solution. No mark was awarded as they did not address the question.*
- *Some students were able to find the number of friends but did not link this number to the question.*

(e) Consider the differential equation **3**

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{1 - \sin x},$$

where $0 \le x < \frac{\pi}{2}$ and $y > 0$.

Given that $y = 1$ when $x = \frac{\pi}{6}$, express y as a function of x.

$$\int \frac{dy}{y} = \int \frac{\cos x}{1-\sin x}\,dx$$

$$\int \frac{dy}{y} = -\int \frac{-\cos x}{1-\sin x}\,dx$$

$\therefore \ \ln|y| = -\ln|1-\sin x| + c$

$\because \ y > 0$, and $0 \le \sin x < 1$

$\therefore \ \ln y = -\ln(1-\sin x) + c$

Substitute $x = \frac{\pi}{6}$, $y = 1$, we get $c = -\ln 2$

That is, $\ln y = -\ln(1-\sin x) - \ln 2$

$$= \ln\left(\frac{1}{2-2\sin x}\right)$$

$\therefore \ y = \dfrac{1}{2-2\sin x}$

Poorly done.

Many students did not provide a reason for getting rid of absolute value, ½ marks deducted.

Common error: $\int \frac{-\cos x}{1-\sin x}\,dx = \ln(1-\sin x)$, *so had* $y = 2-2\sin x$ as their answer.

End of Question 11 Solutions

Section II   Question 12   16 Marks

(a) The function f is defined by $f(x) = \dfrac{2x+a}{x-2}$, where a is a constant, $x > 2$.

(i) Given that f^{-1} exists, state the value that a cannot take, justifying your answer. **2**

$$f(x) = \frac{2(x-2)+4+a}{x-2}$$

$$= 2 + \frac{a+4}{x-2}$$

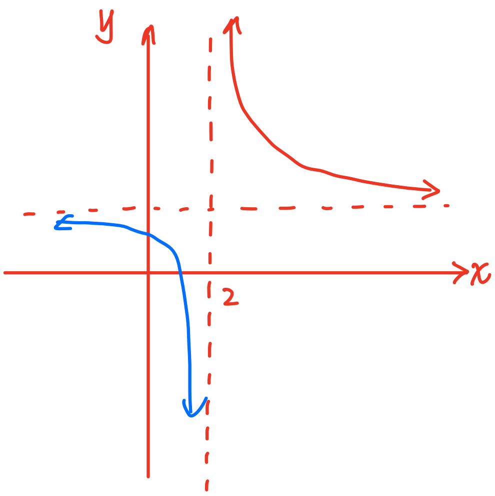
$a+4 \neq 0$ for f^{-1} to exist since $f(x)=2$ fails the HLT.

$\therefore a \neq -4$

Generally well done.

(ii) A function h is said to be self-inverse if $h(x) = h^{-1}(x)$ for all x in the domain of h. **1**

State the range of values of a such that f is a self-inverse function.



For f to be self-inverse, f needs to have symmetry about $y=x$ and be in the same quadrant.

$\therefore$ Blue branch is not self inverse.

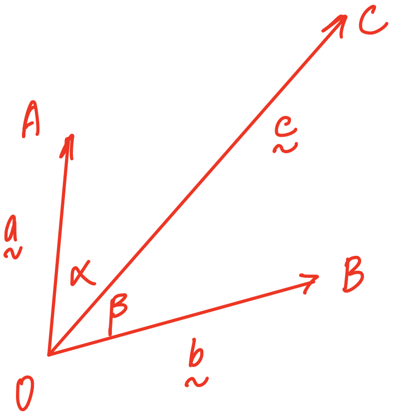
$\therefore a > -4$

Poorly done.

(b) Let $\underset{\sim}{c} = |\underset{\sim}{a}|\underset{\sim}{b} + |\underset{\sim}{b}|\underset{\sim}{a}$, where $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$ are non-zero vectors. **3**

Show that $\underset{\sim}{c}$ bisects the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Method 1



$$\cos\alpha = \frac{\underset{\sim}{a}\cdot\underset{\sim}{c}}{|\underset{\sim}{a}||\underset{\sim}{c}|}$$

$$= \frac{\underset{\sim}{a}\cdot\left(|\underset{\sim}{a}|\underset{\sim}{b} + |\underset{\sim}{b}|\underset{\sim}{a}\right)}{|\underset{\sim}{a}||\underset{\sim}{c}|}$$

$$= \frac{|\underset{\sim}{a}|\,\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{a}|^2|\underset{\sim}{b}|}{|\underset{\sim}{a}||\underset{\sim}{c}|}$$

$$= \frac{\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{a}||\underset{\sim}{b}|}{|\underset{\sim}{c}|}$$

Similarly, $\cos\beta = \dfrac{|\underset{\sim}{a}||\underset{\sim}{b}| + \underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{c}|}$

$\therefore \cos\alpha = \cos\beta$

$\alpha = \beta,\ 2\pi - \beta$

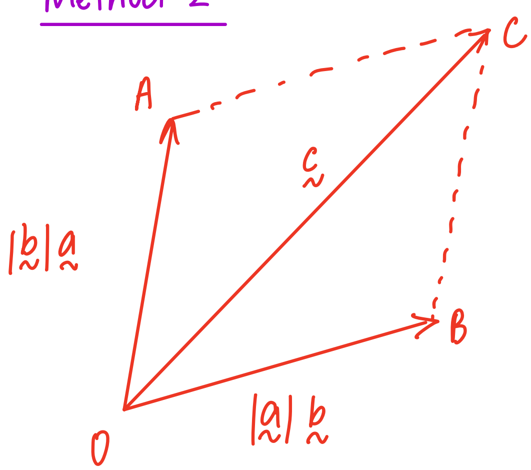
$\because\ 0 \le \alpha, \beta \le \pi$

$\therefore\ \alpha = \beta$ (Either α, β are both acute or obtuse)

Poorly done, students need to explicitly explain why $\alpha = \beta$ from $\cos\alpha = \cos\beta$.

Some students didn't know where to start, the question is asking for angles, so students should recall the dot product.

Method 2



$\underset{\sim}{c} = |\underset{\sim}{a}|\underset{\sim}{b} + |\underset{\sim}{b}|\underset{\sim}{a}$

$\because \left||\underset{\sim}{a}|\underset{\sim}{b}\right| = \left||\underset{\sim}{b}|\underset{\sim}{a}\right|$

$\therefore$ OBCA forms a rhombus with diagonal OC.

Diagonals in a rhombus bisect angles.

$\therefore$ $\underset{\sim}{c}$ bisects the angle between $|\underset{\sim}{a}|\underset{\sim}{b}$ and $\underset{\sim}{a}|\underset{\sim}{b}|$.

$|\underset{\sim}{b}|\underset{\sim}{a}$ is parallel to $\underset{\sim}{a}$, $|\underset{\sim}{a}|\underset{\sim}{b}$ is parallel to $\underset{\sim}{b}$.

$\therefore$ $\underset{\sim}{c}$ bisects the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Quite a few students attempted this method.

A common mistake was diagonals in a parallelogram bisect angles. This is not true, only works for rhombus.

(c) The population of dingoes on an island is modelled by the logistic equation

$$\frac{dy}{dt} = y(1-y),$$

where y is the fraction of the island's carrying capacity of dingoes reached after t years.

Initially, the population of dingoes is estimated to be one-quarter of the island's carrying capacity.

(i) Use the substitution $y = \frac{1}{1-w}$ to transform the logistic equation to $\frac{dw}{dt} = -w$. **2**

$$\frac{dw}{dt} = \frac{dw}{dy} \times \frac{dy}{dt} \qquad \frac{dy}{dw} = \frac{1}{(1-w)^2}$$

$$= (1-w)^2 \times y(1-y)$$

$$= (1-w)^2 \times \frac{1}{1-w}\left(1 - \frac{1}{1-w}\right)$$

$$= (1-w)\left(1 - \frac{1}{1-w}\right)$$

$$= 1 - w - 1$$

$$= -w$$

common mistake : $\frac{dy}{dw} = \frac{-1}{(1-w)^2}$

(ii) 1. Show that $w = Ae^{-t}$ is a solution to $\frac{dw}{dt} = -w$. **1**

$$\frac{dw}{dt} = -Ae^{-t}$$

$$= -w$$

$\therefore$ $w = Ae^{-t}$ is a solution to $\frac{dw}{dt} = -w$.

Quite a few students took the integration approach, this is not recommended considering it's worth 1 mark.

2. Using the solution of $\frac{dw}{dt} = -w$, find the solution of the logistic equation for y satisfying the initial conditions. **2**

$w = Ae^{-t}$

$y = \frac{1}{1-w} = \frac{1}{1-Ae^{-t}}$

$t = 0,\ y = \frac{1}{4}$

$\frac{1}{4} = \frac{1}{1-A}$

$1 - A = 4$

$A = -3$

$\therefore\ y = \frac{1}{1+3e^{-t}}$

Many students failed to see the connection between previous parts and integrated from scratch, this was successful for some, but not so much for others.

(iii) Show when $t = \ln 9$, the dingo population is three-quarters of the island's carrying capacity. **1**

when $t = \ln 9$, $y = \frac{1}{1+3e^{-\ln 9}}$

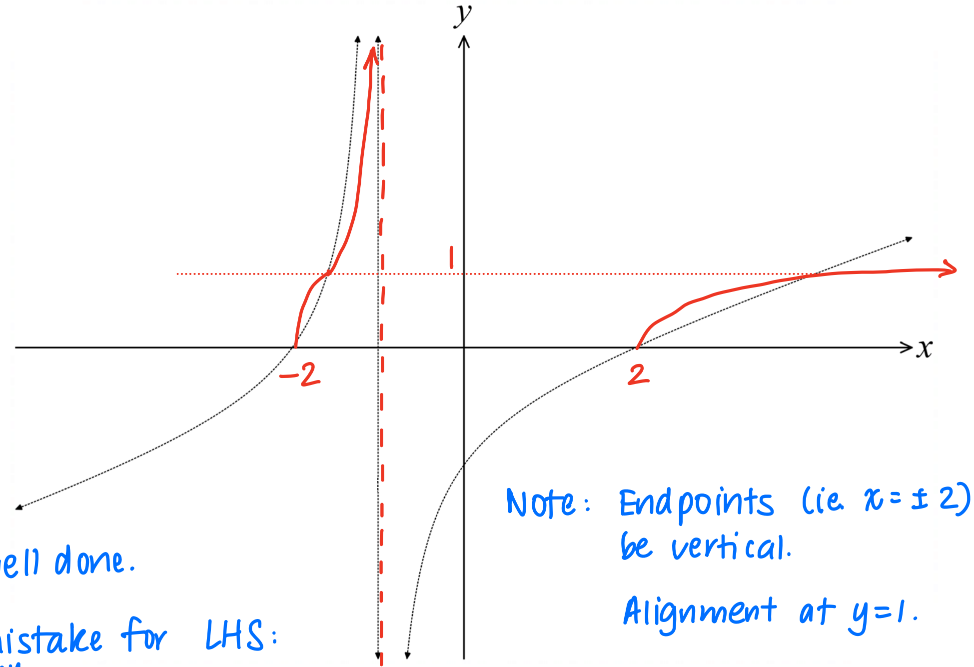
$= \frac{3}{4}$

$\therefore$ when $t = \ln 9$, the dingo population is three quarters of the island's carrying capacity.

Generally well done.

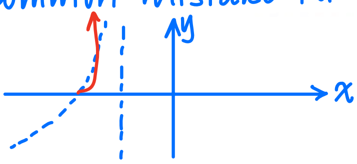
Question 12 (d)

(i) Sketch $y=\sqrt{f(x)}$ **2**



Generally well done.

Common mistake for LHS:

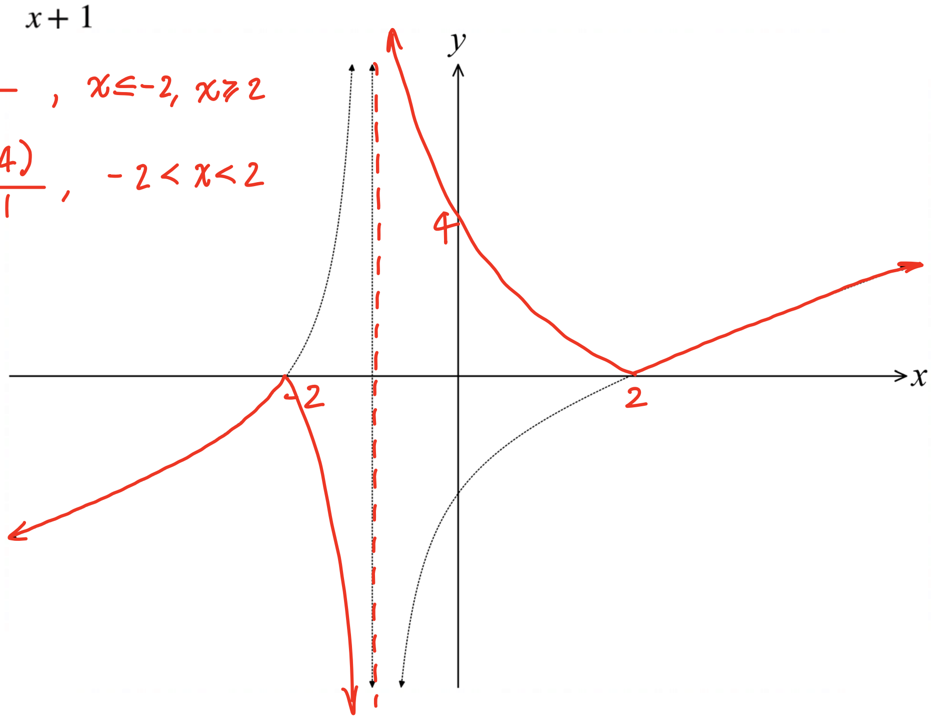


Note: Endpoints (ie $x=\pm 2$) need to be vertical.

Alignment at $y=1$.

(ii) Sketch $y=\dfrac{|x^2-4|}{x+1}$. **2**

$$= \begin{cases} \dfrac{x^2-4}{x+1}, & x \le -2,\ x \ge 2 \\ \dfrac{-(x^2-4)}{x+1}, & -2 < x < 2 \end{cases}$$



Poorly done.

Some students sketched $y=|f(x)|$.

Section II Question 13 Solutions 16 marks

(a) Hugh has six pairs of socks in a drawer, each pair is of a different brand, including his favourite X-men brand.

Each pair consists of two identical socks.

He selects one sock at a time and at random, without replacement.

(i) What is the minimum number of socks he needs to pull out so that he has at least one matching pair socks? **1**

By the pigeonhole principle, 7 socks.

Marking Guideline	Marker's comments
1 mark – Correct answer	- Done well by many candidates.

(a) (continued)

(ii) Find the least number of socks he must select so that the probability of having the X-men pair is higher than the probability of having only 1 X-men sock. **3**

P(both X men socks from n socks) = $\dfrac{{}^{2}C_{2} \times {}^{10}C_{n-2}}{{}^{12}C_{n}}$

P(Only one X men sock from n socks) = $\dfrac{\frac{{}^{2}C_{1}}{2!} \times {}^{10}C_{n-1}}{{}^{12}C_{n}}$

$$\frac{{}^{2}C_{2} \times {}^{10}C_{n-2}}{{}^{12}C_{n}} > \frac{\frac{{}^{2}C_{1}}{2!} \times {}^{10}C_{n-1}}{{}^{12}C_{n}}$$

$${}^{10}C_{n-2} > {}^{10}C_{n-1} \quad \text{as} \quad {}^{12}C_{n} > 0$$

$$\frac{10!}{(10-(n-2))!\,(n-2)!} > \frac{10!}{(10-(n-1))!\,(n-1)!}$$

$$\frac{1}{(12-n)!\,(n-2)!} > \frac{1}{(11-n)!\,(n-1)!}$$

$$(11-n)!\,(n-1)! > (12-n)!\,(n-2)!$$

$$(11-n)!\,(n-1)(n-2)! > (12-n)(11-n)!\,(n-2)!$$

$$\therefore n-1 > 12-n$$

$$2n > 13$$

$$\therefore n > \frac{13}{2}$$

$\therefore n = 7$ is the least number of socks.

Marking Guideline	Marker's comments
3 marks – Correct solution **2 marks** – Substantial progress towards the correct solution. **1 mark** – Identifying the correct expression for one of the probabilities.	- This question was not done well by majority of the candidates. - Many candidates showed limited understanding of the question and were unable to derive the correct probability expression for either the X-men pair of socks or for just one X-men sock. - Some candidates who made substantial progress still made the mistake of not considering that the matching pair of X-men socks were identical when deriving the expression for choosing one X-men sock.

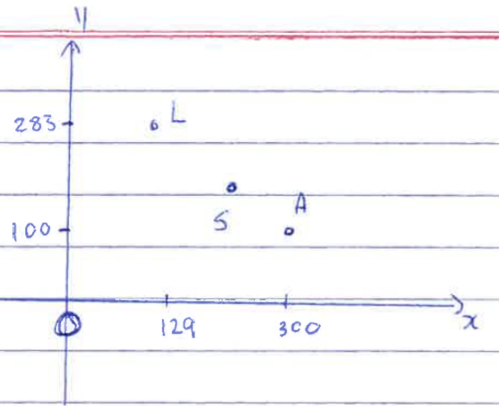
(b) A ship S is travelling with a constant velocity, $\underset{\sim}{v}$ km/h, where

$$\underset{\sim}{v} = \begin{pmatrix} -12 \\ 15 \end{pmatrix}.$$

At time $t = 0$, the ship is at point A (300, 100) relative to an origin O, where distances are measured in kilometres.

A lighthouse is located at a point L (129, 283).

(i) Show that $\overrightarrow{LS} = \begin{pmatrix} 171 \\ -183 \end{pmatrix} + t\begin{pmatrix} -12 \\ 15 \end{pmatrix}$ **2**



$$\underset{\sim}{r}(t) = \int \underset{\sim}{v}\, dt$$

$$= \begin{pmatrix} -12t + C_1 \\ 15t + C_2 \end{pmatrix}$$

$$\text{At } r(0) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 300 \\ 100 \end{pmatrix}$$

$$\therefore r(t) = \begin{pmatrix} -12t + 300 \\ 15t + 100 \end{pmatrix} = \overrightarrow{OS}$$

$$\therefore \overrightarrow{LS} = \overrightarrow{OS} - \overrightarrow{OL}$$

$$= \begin{pmatrix} -12t + 300 \\ 15t + 100 \end{pmatrix} - \begin{pmatrix} 129 \\ 283 \end{pmatrix}$$

$$= \begin{pmatrix} 171 - 12t \\ -183 + 15t \end{pmatrix}$$

$$= \begin{pmatrix} 171 \\ -183 \end{pmatrix} + t\begin{pmatrix} -12 \\ 15 \end{pmatrix}$$

Marking Guideline	Marker's comments
2 marks – Correct Solution **1 mark** – Showing how $\begin{pmatrix} 171 \\ -183 \end{pmatrix}$ is derived.	- As this is a 'SHOW' question, candidates were penalised if they did not demonstrate how the result $t\begin{pmatrix} -12 \\ 15 \end{pmatrix}$ was derived. Integration (as shown in the solution) or explicitly stating that displacement is the product of velocity and time is required, so markers do not assume the candidate has merely taken the result from the question without proper justification.

(b) **(continued)**

(ii) By finding $\left|\overrightarrow{LS}\right|$, find the value of t when the ship is closest to the lighthouse. **2**

$$|\overrightarrow{LS}| = \left| \begin{matrix} 171 - 12t \\ -183 + 15t \end{matrix} \right|$$

$$= \sqrt{(171-12t)^2 + (-183+15t)^2}$$

$$= \sqrt{(171^2 - 171 \times 2 \times 12t + (12t)^2) + ((-183)^2 - 2 \times 183 \times 15t + (15t)^2)}$$

$$= \sqrt{369t^2 - 9594t + 62730}$$

$$= \sqrt{369} \times \sqrt{t^2 - 26t + 170}$$

$$= 3\sqrt{41}\,(t^2 - 26t + 170)^{1/2}$$

$$\therefore \frac{d|\overrightarrow{LS}|}{dt} = 3\sqrt{41} \times \frac{1}{2} \times (t^2 - 26t + 170)^{-1/2} \times (2t - 26)$$

$$= \frac{3\sqrt{41}\,(t-13)}{(t^2 - 26t + 170)^{1/2}}$$

let $\frac{d|\overrightarrow{LS}|}{dt} = 0$

$$0 = \frac{3\sqrt{41}\,(t-13)}{(t^2 - 26t + 170)^{1/2}}$$

$\therefore t - 13 = 0$

$t = 13$

Test $t = 13$

t	12	13	14
$\frac{d\lvert\overrightarrow{LS}\rvert}{dt}$	-13.58	0	13.58
	$\searrow$	—	$\nearrow$

$\therefore$ $t = 13$ is a minimum ('global) turning point.

$\therefore$ At $t = 13$ hours, is when the ship is closest to the lighthouse.

Marking Guideline	Marker's comments
2 marks – Correct Solution **1 mark** – Correct working to obtain $\left\|\overrightarrow{LS}\right\| = \sqrt{369t^2 - 9594t + 62730}$	- A significant number of candidates were not able to derive the correct expression for $\left\|\overrightarrow{LS}\right\|$ or made careless errors in the arithmetic calculations. Candidates need to take more care. - Candidates were penalised if no justification were explicitly given to why result of $t = 13$ hours should give the closest distance between ship and the lighthouse.

(b) **(continued)**

(iii) An alarm will sound if the ship travels within 20 km of the light house. State whether the alarm will sound. Justify your answer. **1**

When $t = 13$

$|\overrightarrow{LS}| = \sqrt{369(13^2 - 26(13) + 170}$

$= \sqrt{369}$

$< \sqrt{400}$

$= 20\text{Km}$

$\therefore$ The alarm will sound

Marking Guideline	Marker's comments
1 mark – Correct answer with justification	- Candidates were penalised if no (or incorrect) justification were provided.

(c) A vertical wall, height H metres, stands on horizontal ground.

An object is thrown towards the wall and is projected with an initial speed u m/s at an angle θ with the horizontal plane.

The object is starts from a point on the ground D metres from the wall and it just clears the wall at the highest point of its path.

Let the position of the object at time t be given by $\mathbf{r}(t)$ where

$$\mathbf{r}(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$$ (Do NOT prove this.)

(i) Show that the particle reaches the highest point on its path when $t = \dfrac{u\sin\theta}{g}$. **2**

$r(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$

$\dot{r}(t) = \begin{pmatrix} u\cos\theta \\ u\sin\theta - gt \end{pmatrix}$

Max height is reached when $u\sin\theta - gt = 0$

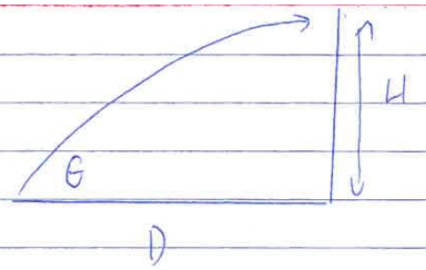
$\therefore gt = u\sin\theta$

$t = \dfrac{u\sin\theta}{g}$

Marking Guideline	Marker's comments
2 marks – Correct Solution **1 mark** – Correct derivative of $y = ut\sin\theta - \frac{1}{2}gt^2$	- Done well by many candidates.

(c) **(continued)**

(ii) Show that the speed of projection is given by $u^2 = \dfrac{g}{2H}\left(4H^2 + D^2\right)$. **3**



At $t = \dfrac{u\sin\theta}{g}$ $\quad x = D$ and $y = H$

$\therefore \; D = ut\cos\theta$

$= u\left(\dfrac{u\sin\theta}{g}\right)\cos\theta$

$D = \dfrac{u^2\sin\theta\cos\theta}{g}$ $\quad (1)$

$H = ut\sin\theta - \frac{1}{2}gt^2$

$H = u\left(\dfrac{u\sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\dfrac{u\sin\theta}{g}\right)^2$

$= \dfrac{u^2\sin^2\theta}{g} - \dfrac{u^2\sin^2\theta}{2g}$

$= \dfrac{2u^2\sin^2\theta - u^2\sin^2\theta}{2g}$

$H = \dfrac{u^2\sin^2\theta}{2g}$ $\quad (2)$

$\therefore \; RHS = \dfrac{g}{2H}\left(4H^2 + D^2\right)$

$= \dfrac{g}{2\left(\frac{u^2\sin^2\theta}{2g}\right)}\left(4\times\left(\dfrac{u^2\sin^2\theta}{2g}\right)^2 + \left(\dfrac{u^2\sin\theta\cos\theta}{g}\right)^2\right)$

$= \dfrac{g^2}{u^2\sin^2\theta}\left(\dfrac{u^4\sin^4\theta}{g^2} + \dfrac{u^4\sin^2\theta\cos^2\theta}{g^2}\right)$

$= \dfrac{g^2}{u^2\sin^2\theta}\left(\dfrac{u^4\sin^2\theta}{g^2}\left(\sin^2\theta + \cos^2\theta\right)\right)$

$= \dfrac{g^2}{u^2\sin^2\theta}\left(\dfrac{u^4\sin^2\theta}{g^2}\right)$

$= u^2$

$= LHS$

(c) (ii) (continued)

Marking Guideline	Marker's comments
3 marks – Correct solution **2 marks** – Correct expression for *D* and *H* in terms of u, g and θ. **1 mark** – Correct expression for either D or H in terms u, g and θ.	- This question was not done well by many candidates. - A significant number of candidates were not able to correctly derive the expression for *H* or *D* or both, and instead incorrectly manipulated their working to the desired result. Candidates need to take more care with their working and if their solution does not achieve the correct result than they should check their working rather fudge their working.

(c) (continued)

(iii) Show that θ, the angle of projection, is given by **2**

$$\theta = \tan^{-1}\left(\frac{2H}{D}\right).$$

From (ii)

$H = \dfrac{u^2 \sin^2\theta}{2g}$

$\therefore 2H = \dfrac{u^2 \sin^2\theta}{g}$

$D = \dfrac{u^2 \sin\theta \cos\theta}{g}$

$\therefore \dfrac{2H}{D} = \dfrac{\dfrac{u^2 \sin^2\theta}{g}}{\dfrac{u^2 \sin\theta \cos\theta}{g}}$

$= \dfrac{\sin\theta}{\cos\theta}$

$\therefore \operatorname{Tan}\theta = \dfrac{2H}{D}$

$\therefore \theta = \operatorname{Tan}^{-1}\left(\dfrac{2H}{D}\right)$

Marking Guideline	Marker's comments
2 marks – Correct Solution **1 mark** – Significant progress towards correct solution.	- This question was not well done by many candidates. A significant number of candidates started from the result and worked backward, produce working that did not justify why the result is true. The goal of a proof is to show that a statement is true through logical deduction, not to assume its truth and work from there. No marks were awarded for such effort. - Some candidates had incorrect working and then incorrectly manipulated their working to fit the result to be proven. Candidates need to be aware that marks will be not be awarded for incorrect working.

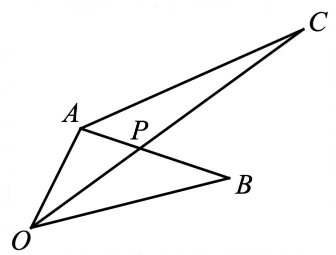
Question 14 — 14 marks

(a) With respect to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.

Point P is on the line AB such that $AP : PB = m : n$, where m and n are positive integers.

Point C is on OP extended such that $OP : PC = 1 : 2$.

(i) Show that $\overrightarrow{AC} = \left(\dfrac{2n-m}{m+n}\right)\mathbf{a} + \left(\dfrac{3m}{m+n}\right)\mathbf{b}$ **3**



$AP : PB = m : n$

$\therefore \overrightarrow{AP} = \dfrac{m}{m+n}(\mathbf{b}-\mathbf{a})$

$\therefore \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \mathbf{a} + \dfrac{m}{m+n}(\mathbf{b}-\mathbf{a})$

As $\overrightarrow{PC} = 2\,\overrightarrow{OP}$ and $\overrightarrow{AC} = \overrightarrow{AP} + \overrightarrow{PC}$ then:

$$\overrightarrow{AC} = \frac{m}{m+n}(\mathbf{b}-\mathbf{a}) + 2\left(\mathbf{a} + \frac{m}{m+n}(\mathbf{b}-\mathbf{a})\right)$$

$$= \left(2 - 3\times\frac{m}{m+n}\right)\mathbf{a} + \left(3\times\frac{m}{m+n}\right)\mathbf{b}$$

$$= \left(\frac{2(m+n)-3m}{m+n}\right)\mathbf{a} + \left(\frac{3m}{m+n}\right)\mathbf{b}$$

$$= \left(\frac{2n-m}{m+n}\right)\mathbf{a} + \left(\frac{3m}{m+n}\right)\mathbf{b}$$

Question 14 (continued)

(a) (i) **Method 2:** $\overrightarrow{AC} = -\overrightarrow{OA} + \overrightarrow{OC}$ and $\overrightarrow{OC} = 3\,\overrightarrow{OP}$

As $\overrightarrow{OC} = 3\,\overrightarrow{OP}$ then:

$$\overrightarrow{AC} = -\mathbf{a} + 3\left(\mathbf{a} + \frac{m}{m+n}(\mathbf{b}-\mathbf{a})\right)$$

$$= -\mathbf{a} + \left(3 - \frac{3m}{m+n}\right)\mathbf{a} + \left(3 \times \frac{m}{m+n}\right)\mathbf{b}$$

$$= \left(2 - \frac{m}{m+n}\right)\mathbf{a} + \left(3 \times \frac{m}{m+n}\right)\mathbf{b}$$

$$= \left(\frac{2(m+n) - 3m}{m+n}\right)\mathbf{a} + \left(\frac{3m}{m+n}\right)\mathbf{b}$$

$$= \left(\frac{2n-m}{m+n}\right)\mathbf{a} + \left(\frac{3m}{m+n}\right)\mathbf{b}$$

Comment: It was pleasing that most students could derive $\overrightarrow{OP}$.

A common mistake was having C in the interior of the triangle.

Some students couldn't achieve full marks as they left the marker to believe they could handle the algebra and just stated, too early, the result they had to show.

(ii) Find the ratio $AP : PB$ such that AC is parallel to OB. **1**

The coefficient of **a** must be zero

$\therefore\ 2n = m$

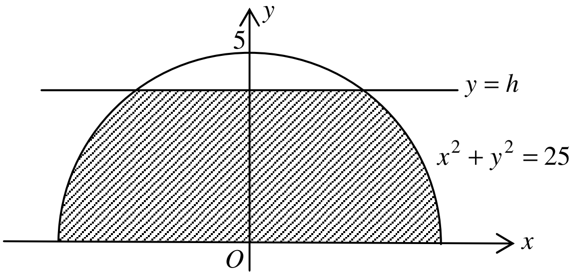
$\therefore\ m : n = 2 : 1$

Comment: Most students could derive the result $2n - m = 0$.

Some incorrectly believed that to be parallel to OB that AC had to equal OB.

Question 14 (continued)

(b) (i) **2**



In the figure above, the shaded region enclosed by the circle $x^2 + y^2 = 25$, the x-axis, and the straight line $y = h$ (where $0 \le h \le 5$) is revolved about the y-axis.

Show that the volume of the solid of revolution is $\left(25h - \dfrac{h^3}{3}\right)\pi$ cubic units.

$$x^2 = 25 - y^2$$

$$\text{Vol} = \pi \int_0^h x^2 \, dy$$

$$= \pi \int 25 - y^2 \, dy$$

$$= \pi \left[25y - \frac{1}{3}y^3 \right]_0^h$$

$$= \pi \left(25h - \frac{1}{3}h^3 - 0 \right)$$

$$= \pi \left(25h - \frac{1}{3}h^3 \right) \text{ cu}$$

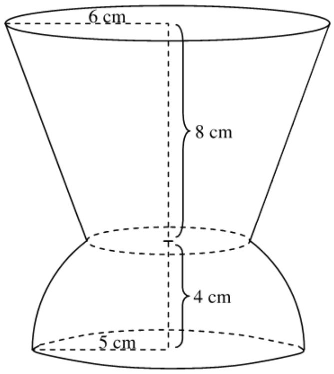
Comment: This was done very well.

Surprisingly, some students wanted to test themselves by doing this indirectly.

Question 14 (continued)

(b) (continued)

In the diagram below, an empty coffee cup consists of two parts.
The lower part is in the shape of the solid described in (b) (i).
The upper part is a frustum of a circular cone of height 8 cm, where the radius of the top of the cup is 6 cm.
Hot coffee is poured into the cup to a depth h cm at a rate of 8 cm^3/s, where $0 \le h \le 12$.
Let V cm^3 be the volume of coffee in the cup.



(ii) Find the rate of increase of the depth of coffee when the depth is 3 cm. **2**

For $0 \le h \le 4$, $V = \left(25h - \dfrac{h^3}{3}\right)\pi$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$8 = \left(25 - h^2\right)\pi \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{8}{\left(25 - h^2\right)\pi}$$

At $h = 3$: $\dfrac{dh}{dt} = \dfrac{8}{(25-3^2)\,\pi} = \dfrac{1}{2\pi}$ cm/s

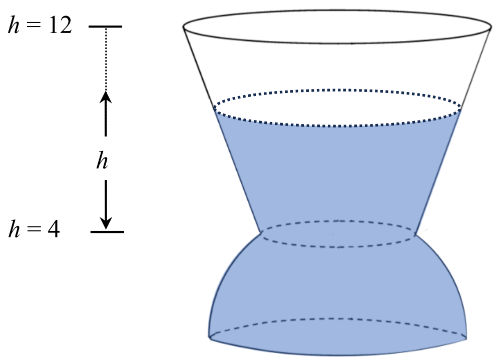
i.e. the rate of increase of the depth of coffee is $\dfrac{1}{2\pi}$ cm s^{-1} .

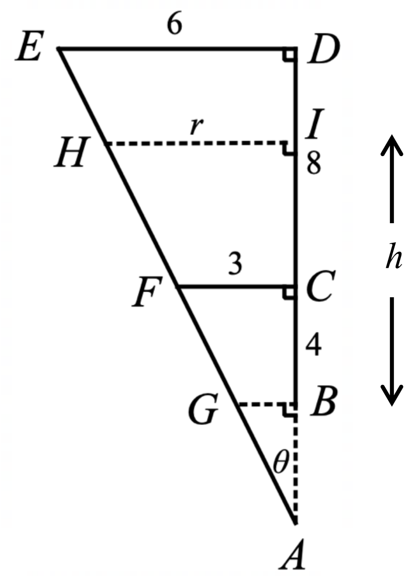
Comment: This was well done by most students.

A common mistake was leaving out π or forgetting to substitute $h = 3$.

(b) (iii) Show that the volume of coffee in the cup when for $4 \le h \le 12$ is given by **3**

$$V = \frac{164\pi}{3} + \frac{3\pi}{64}(h+4)^3.$$





The volume of the coffee for $0 \le h \le 4$ is found by substituting $h = 4$ into result of b (i) i.e. $\frac{236\pi}{3}$ cu.

The radius of a circle at height h cm is r cm.
As $\Delta\, ACF \parallel\!| \, \Delta\, ADE$ then $AC = 8$ and so $AB = 4$.

Method 1a: Let θ be the semi-apex angle.

$$\tan\theta = \frac{6}{16} = \frac{r}{h+4}$$

$$\therefore r = \frac{3}{8}(h+4)$$

The frustum's larger cone has a height of $(h + 4)$ and radius r.
The smaller cone had a height of 8 and radius 3.

$$V = \frac{\pi}{3}\left(r^2(h+4) - 3^2 \times 8\right)$$

$$= \frac{\pi}{3}\left(\frac{9}{64}(h+4)^3 - 3^2 \times 8\right)$$

$$\text{Volume of coffee} = \frac{236\pi}{3} + \frac{3\pi}{64}(h+4)^3 - 24\pi$$

$$= \frac{164\pi}{3} + \frac{3\pi}{64}(h+4)^3$$

Question 14 (continued)

(b) (iii) (continued)

Method 1b: Similarity

Basically same as Method 1a

$\Delta\, AIH \mathbin{|||} ACF$ (AA)

$$\therefore \frac{r}{h+4} = \frac{3}{8}$$

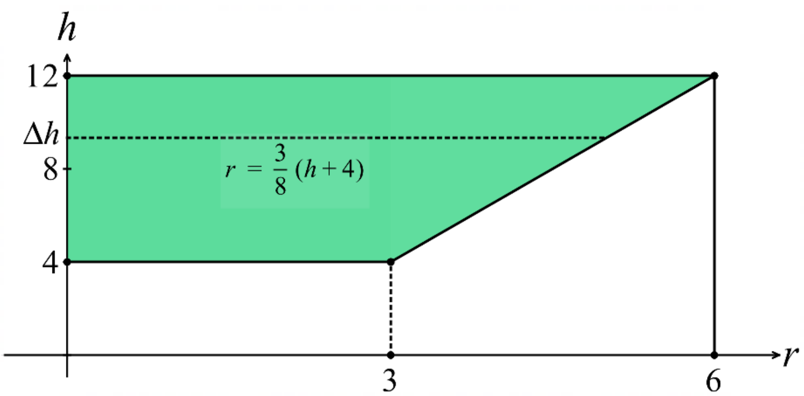
$$\therefore r = \frac{3}{8}(h+4)$$

The rest finishes as Method 1a.

Method 2: Volume of revolution

We know: $h = 4, r = 3$ and $h = 12, r = 6$
Rotating the shaded region will give the volume of the frustum.

The equation of the line is $h = \frac{8}{3}r - 4$ or $r = \frac{3}{8}(h+4)$.



$$\text{Vol} = \pi\int_4^h r^2\, dh$$

$$= \pi\int_4^h \frac{9}{64}(h+4)^2\, dh$$

$$= \frac{9\pi}{64}\left[\frac{1}{3}(h+4)^3\right]_4^h$$

$$= \frac{3\pi}{64}\left[(h+4)^3 - 8^3\right]$$

Stated earlier, the amount of coffee at $h = 4$ is $\frac{236\pi}{3}$ $\left[h = 4 : \left(25h - \frac{h^3}{3}\right)\pi\right]$

Total volume of coffee at height $h = \frac{3\pi}{64}(h+4)^3 - 24\pi + \frac{236\pi}{3} = \frac{164\pi}{3} + \frac{3\pi}{64}(h+4)^3$

Comment: This was not done well by the majority of students.

Surprisingly, students thought they could just fudge their way through this problem. The term '$h + 4$' was introduced without justification, probably through 'reverse engineering'.
A common mistake was thinking the triangle was only 12 metres high, and not 16 m.

Even though students were more successful with the integration approach, there was no attempt to defined the variables x and y.

(b) (iv) After the cup is fully filled, it cracks at the bottom. **3**
The coffee leaks at a rate of 2 cm^3/s.

Find the rate of decrease of the depth of coffee after 15 seconds of leaking.
Leave your answer correct to 3 significant figures.

After 15 seconds: Volume of coffee = $\frac{164\pi}{3} + \frac{3\pi}{64}(12+4)^3 - 2\times 15$

What is h?

$$\frac{164\pi}{3} + \frac{3\pi}{64}(h+4)^3 - 2\times 15 = \frac{164\pi}{3} + \frac{3\pi}{64}(12+4)^3 - 2\times 15$$

$$\therefore \frac{3\pi}{64}(h+4)^3 = 192\pi - 30$$

$$\therefore h+4 = 4\left(\frac{64\pi - 10}{\pi}\right)^{\frac{1}{3}}$$

$$\therefore h \approx 11.73 \ (>4)$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{9\pi}{64}(h+4)^2 \frac{\mathrm{d}h}{\mathrm{d}t}$$

After 15 seconds, $-2 = \frac{9\pi}{64}\left[4\left(\frac{64\pi - 10}{\pi}\right)^{\frac{1}{3}}\right]^2 \frac{\mathrm{d}h}{\mathrm{d}t}$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{-8}{9\pi^{\frac{1}{3}}(64\pi - 10)^{\frac{2}{3}}}$$
$$\approx -0.0183$$

i.e. the rate of decrease of the depth of coffee is 0.0183 cm s^{-1} .

Comment Most students could achieve some sort of expression for $\frac{dV}{dt}$, but this only had a minor mark attached to it.

Quite a few students had completely unreasonable rates. This would have indicated a mistake has occurred.

End of solutions